Intro Video: Section 5.1
Areas and Distances

Math F251X: Calculus I

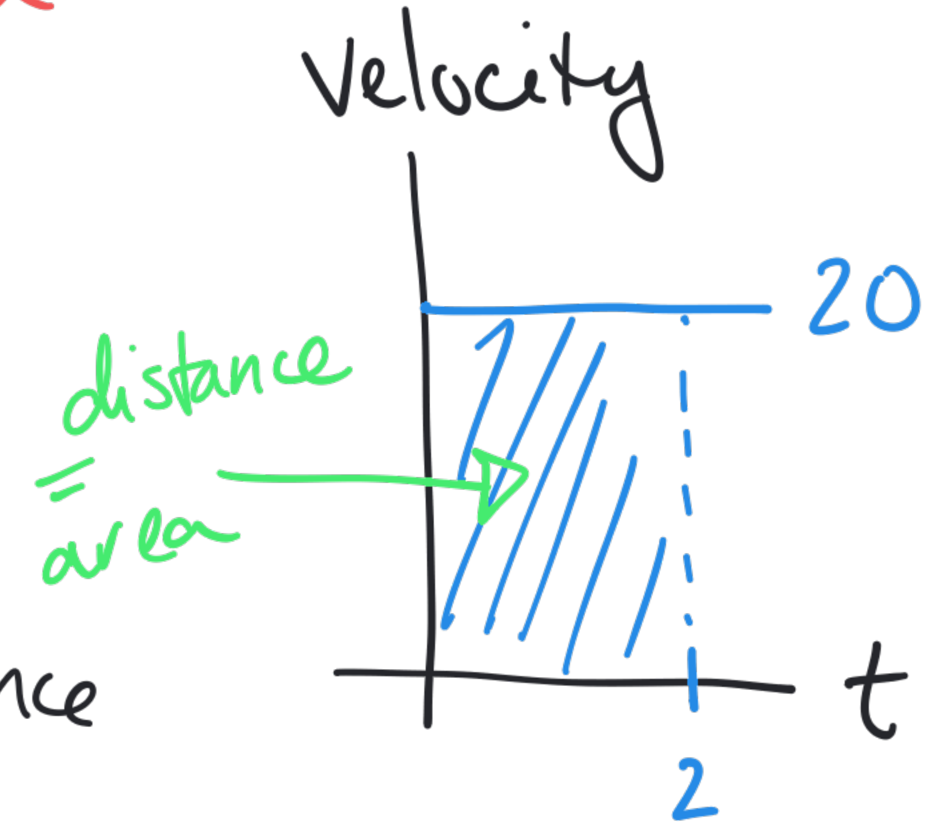
New idea: Calculating <u>area</u>.

Two Big ideas of calculus:

① Rates of Change

$\rightarrow$ derivatives!

② Area under a curve / accumulation of area



Idea: distance = rate × time!

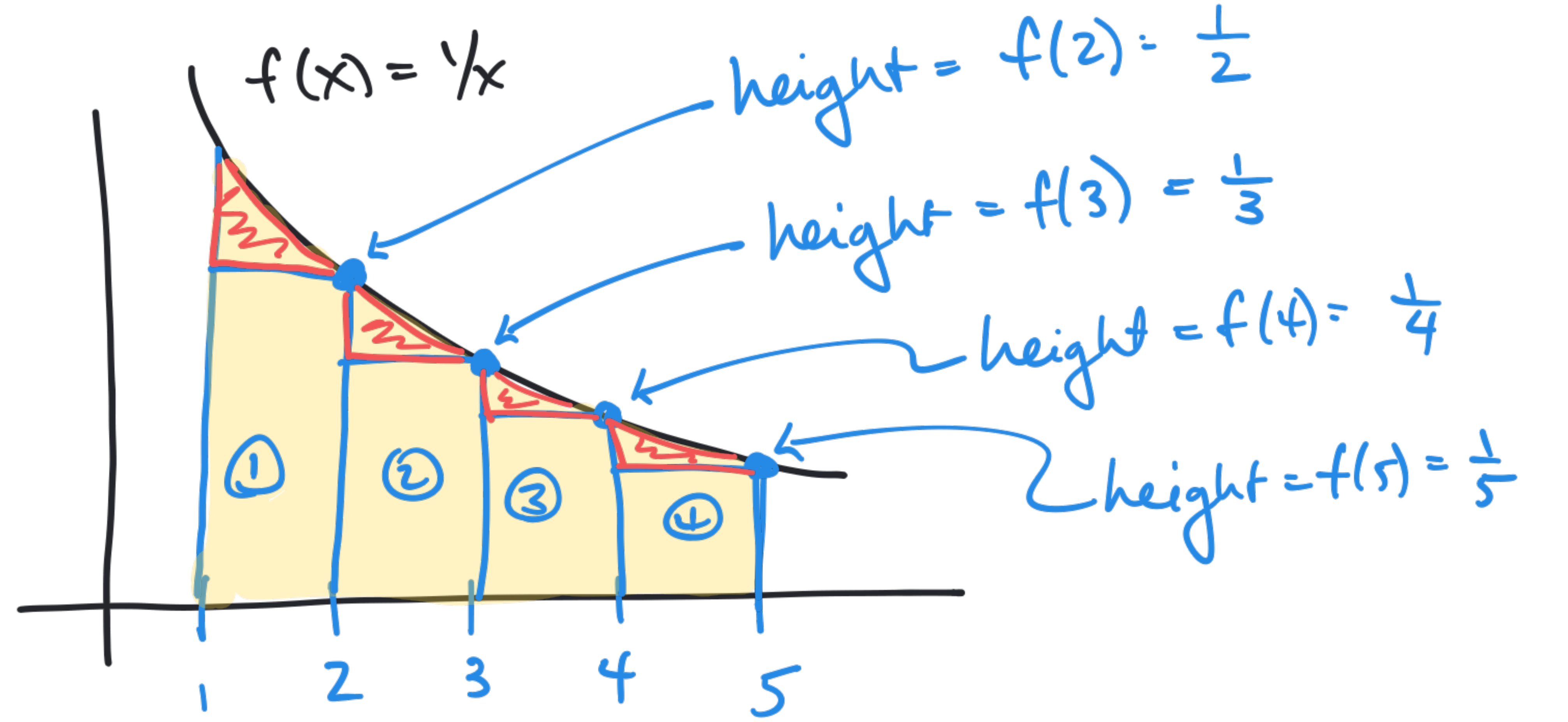
$\rightarrow$ If you travel for 2 hours at a rate of 20 miles/hour, then the total distance is $2(20) = 40$ miles!

How do we find the area under a curve?

Example: Approximate the area bounded between $f(x) = \frac{1}{x}$, $x = 1$, $x = 5$, and the x-axis.

Use four rectangles to approximate the area!

→ width $= \frac{5-1}{4} = \frac{4}{4} = 1.$



Total area ≈ area of all rectangles

$= \text{area } 1 + \text{area } 2 + \text{area } 3 + \text{area } 4$

$= 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{3}\right) + 1\left(\frac{1}{4}\right) + 1\left(\frac{1}{5}\right)$

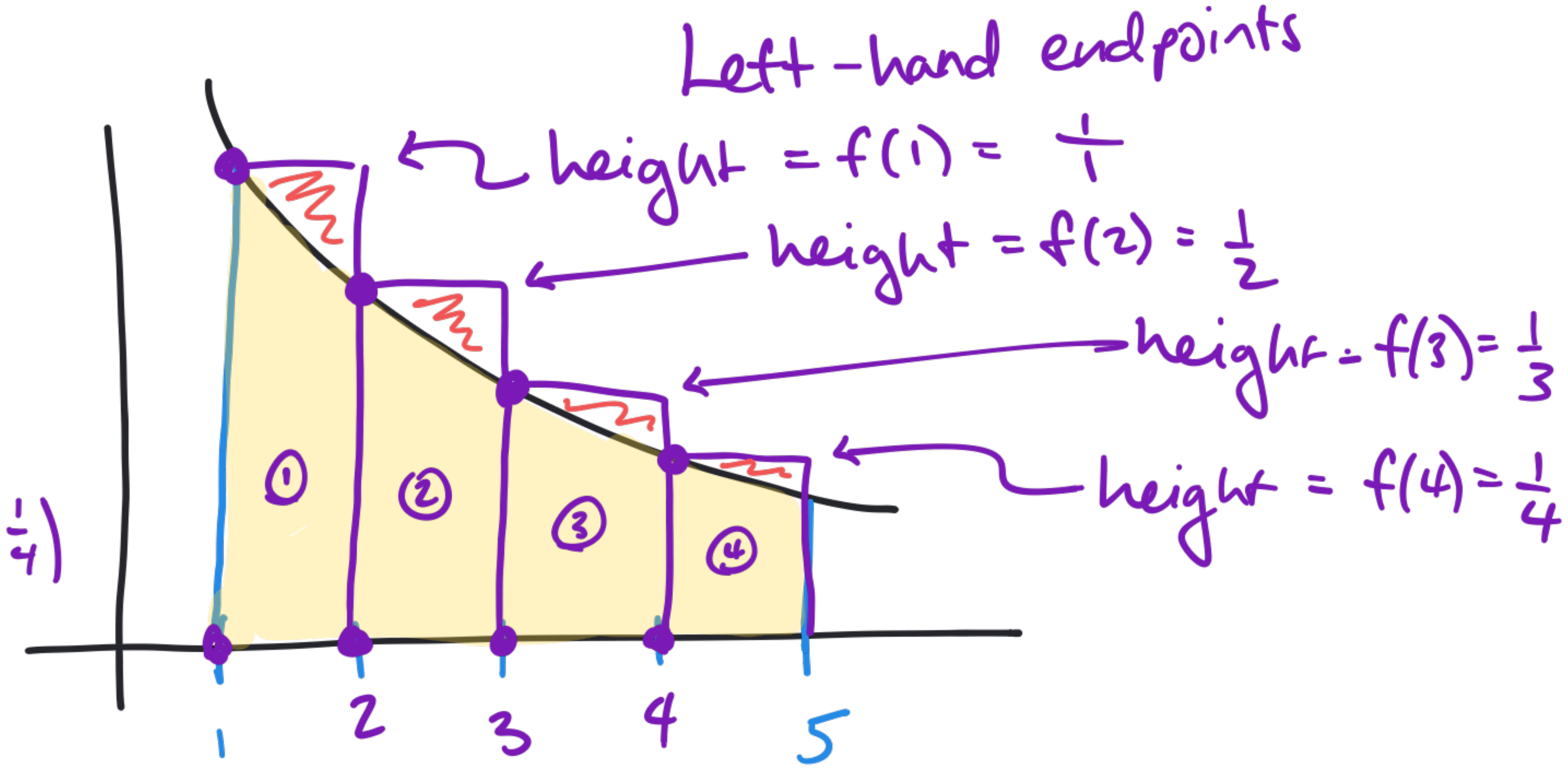
$= \frac{30}{60} + \frac{20}{60} + \frac{15}{60} + \frac{12}{60} = \frac{77}{60}$

How do we find the area under a curve?

Example: Approximate the area bounded between $f(x) = \frac{1}{x}$, $x = 1$, $x = 5$, and the x-axis.

Width $= \frac{5-1}{4} = 1$

height = left-hand endpoints



Area $= 1(1) + 1\left(\frac{1}{2}\right) + 1\left(\frac{1}{3}\right) + 1\left(\frac{1}{4}\right)$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$= \frac{12}{12} + \frac{6}{12} + \frac{4}{12} + \frac{3}{12}$

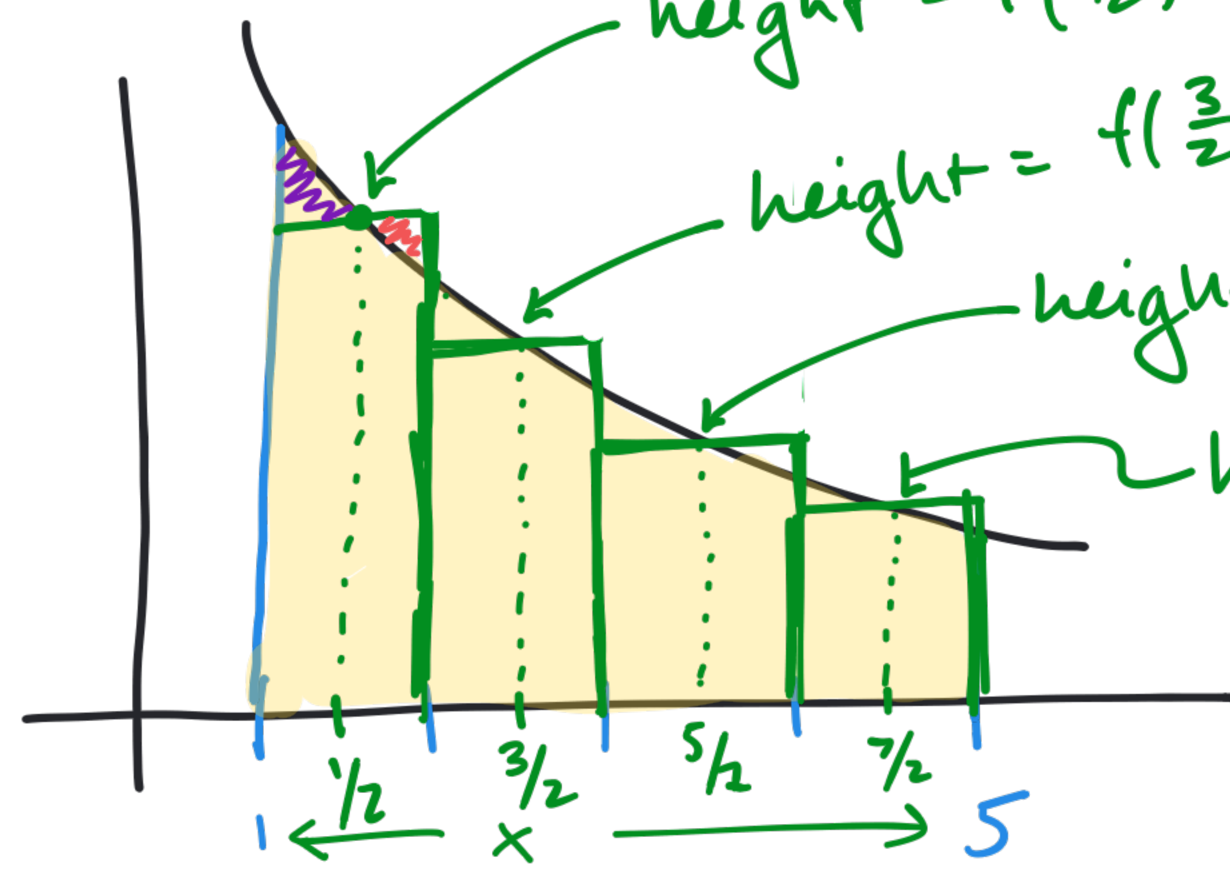
$= \frac{35}{12}$

How do we find the area under a curve?

Example: Approximate the area bounded between $f(x) = \frac{1}{x}$, $x = 1$, $x = 5$, and the x-axis.

Width $= \frac{5-1}{4} = \frac{4}{4} = 1$

Use midpoints to determine height!



height $= f(1/2) = 2$

height $= f(\frac{3}{2}) = \frac{2}{3}$

height $= f(\frac{5}{2}) = \frac{2}{5}$

height $= f(\frac{7}{2}) = \frac{2}{7}$

Area $= 1(2) + 1(\frac{2}{3}) + 1(\frac{2}{5}) + 1(\frac{2}{7})$

$= \frac{352}{105}$

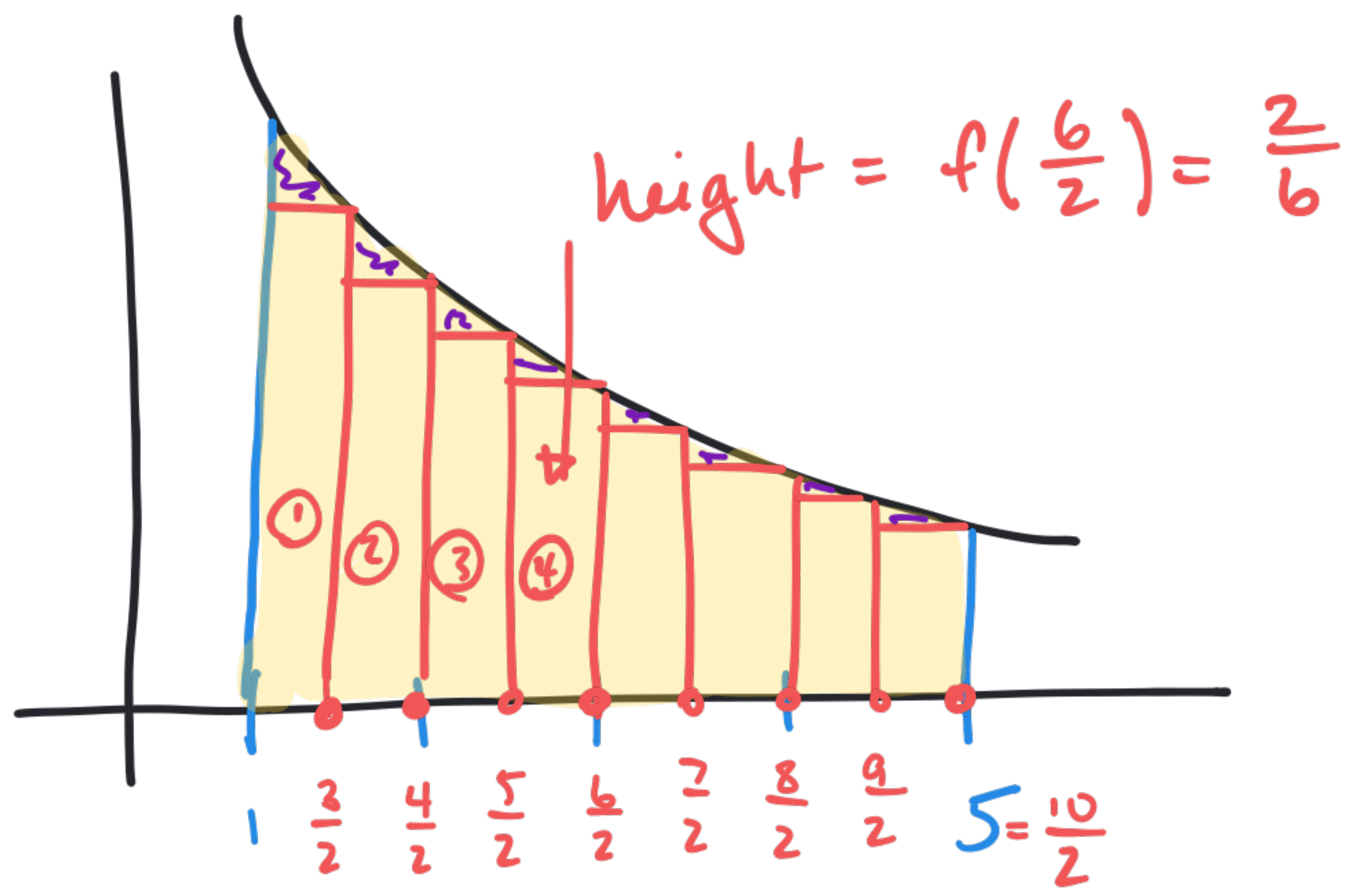
How do we find the area under a curve?

Example: Approximate the area bounded between $f(x) = \frac{1}{x}$, $x = 1$, $x = 5$, and the x-axis.

With 8 rectangles:

$$\text{Width} = \frac{5-1}{8} = \frac{4}{8} = \frac{1}{2}$$

height = value of the function



$$\text{area} = \tfrac{1}{2} f(\tfrac{3}{2}) + \tfrac{1}{2} f(\tfrac{4}{2}) + \tfrac{1}{2} f(\tfrac{5}{2}) + \tfrac{1}{2} f(\tfrac{6}{2}) + \tfrac{1}{2} f(\tfrac{7}{2}) + \tfrac{1}{2} f(\tfrac{8}{2}) + \tfrac{1}{2} f(\tfrac{9}{2}) + \tfrac{1}{2} f(\tfrac{10}{2})$$

$$= \tfrac{1}{2}(\tfrac{2}{3}) + \tfrac{1}{2}(\tfrac{2}{4}) + \tfrac{1}{2}(\tfrac{2}{5}) + \tfrac{1}{2}(\tfrac{2}{6}) + \tfrac{1}{2}(\tfrac{2}{7}) + \tfrac{1}{2}(\tfrac{2}{8}) + \tfrac{1}{2}(\tfrac{2}{9}) + \tfrac{1}{2}(\tfrac{2}{10})$$

$$= \frac{3601}{2520} \approx 1.43$$

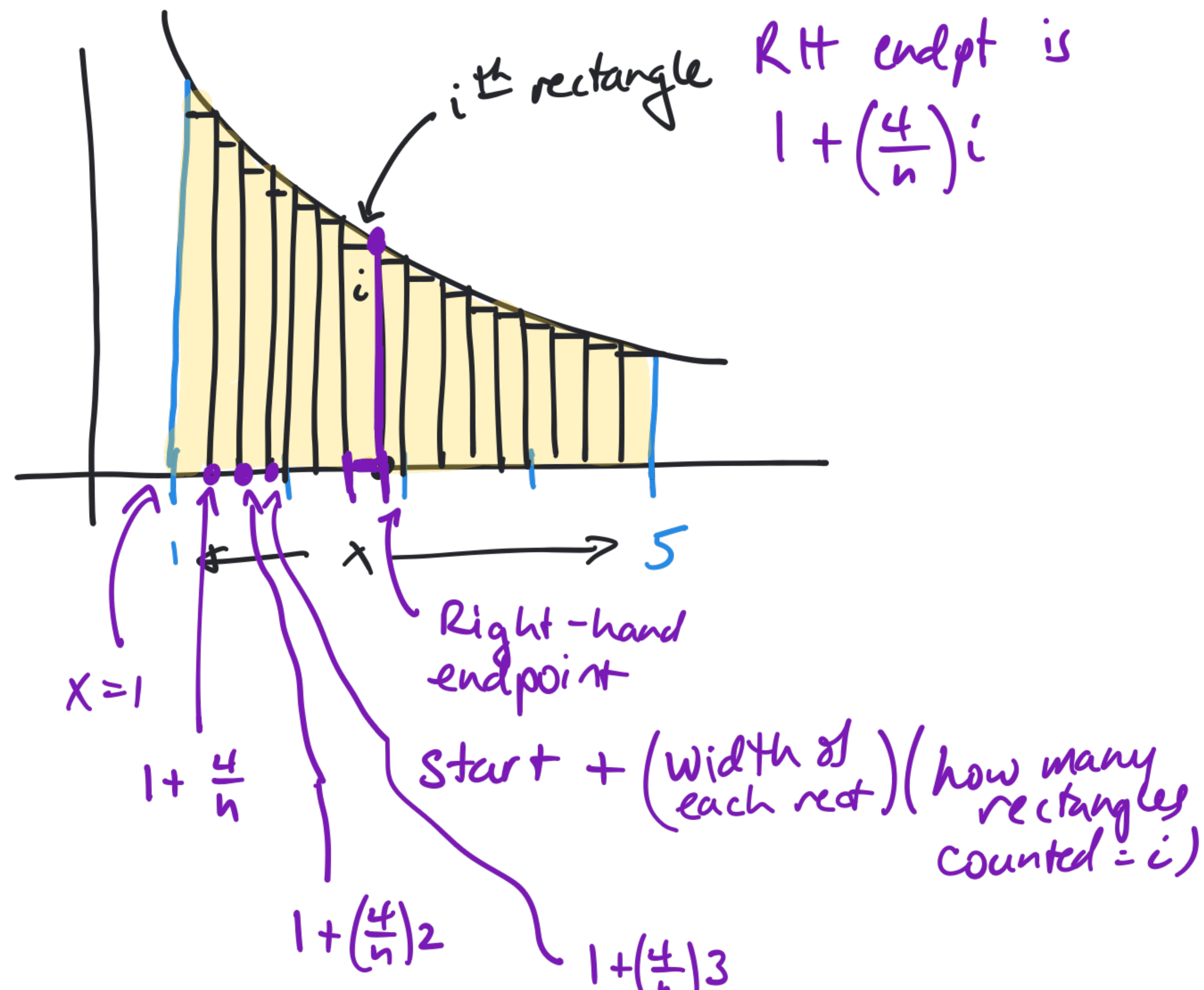
How do we find the area under a curve?

Example: Approximate the area bounded between $f(x) = \frac{1}{x}$, $x = 1$, $x = 5$, and the x-axis.

Cut area into n rectangles.

Width of each rectangle is $\frac{5-1}{n} = \frac{4}{n}$.

i^{th} rectangle has RH endpoint equal to $1 + \left(\frac{4}{n}\right)i$



$$\text{Area} = \sum_{i=1}^{n} \Big(\underbrace{f\left(1 + \left(\tfrac{4}{n}\right)i\right)}_{\text{height}} \cdot \underbrace{\tfrac{4}{n}}_{\text{width}} \Big)$$

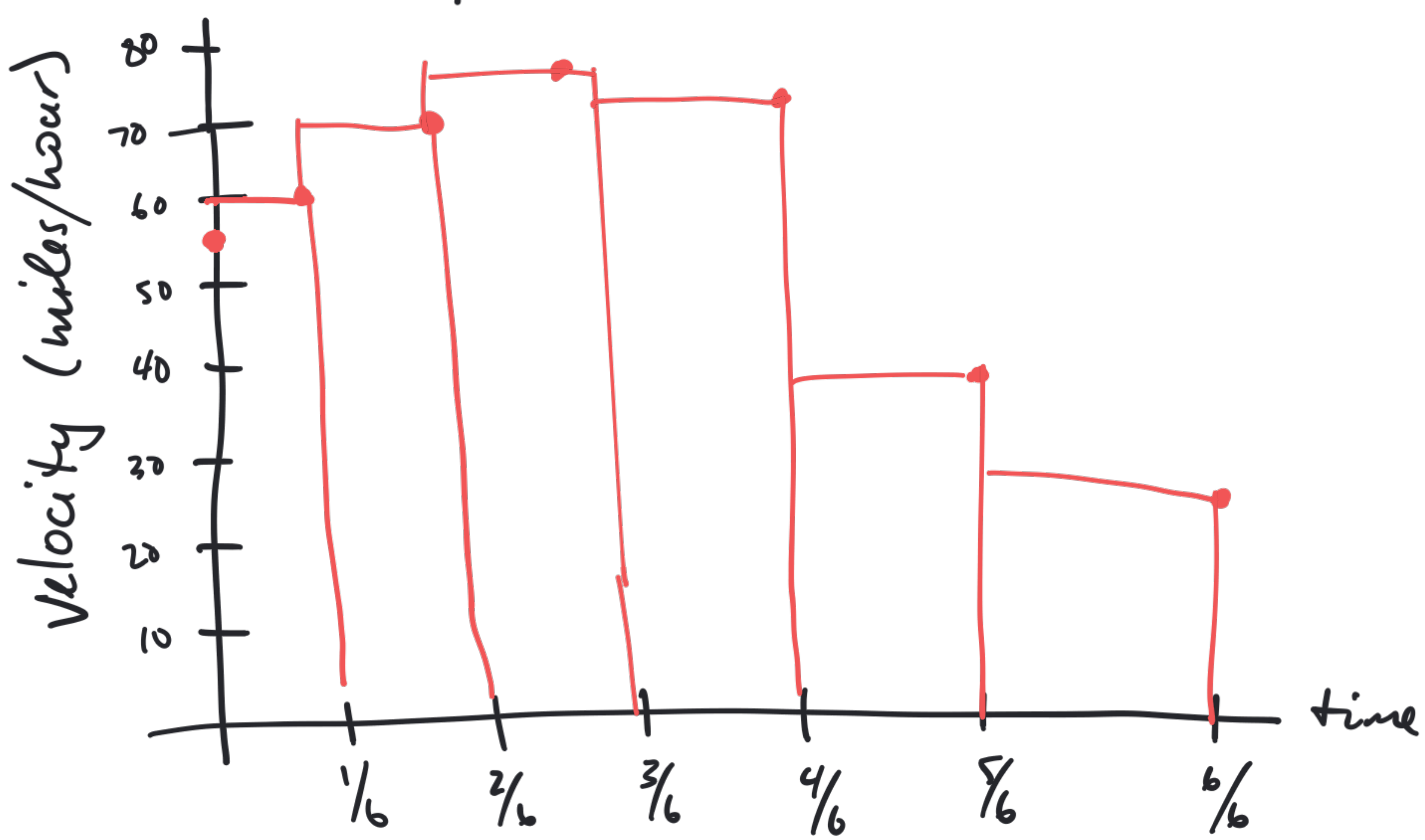
Example: $\sum_{i=1}^{3} f\left(1+\left(\frac{4}{n}\right)i\right)\frac{4}{n} = f\left(1+\left(\frac{4}{n}\right)1\right)\left(\frac{4}{n}\right) + f\left(1+\frac{4}{n}\cdot 2\right)\frac{4}{n} + f\left(1+\frac{4}{n}\cdot 3\right)\frac{4}{n}$

Back to thinking about distance

Suppose we measure velocity at fixed intervals:

every 10 minutes $= \frac{10}{60} = \frac{1}{6}$ of an hour:

	0	1/6	2/6	3/6	4/6	5/6	6/6	hours
time	0	10	20	30	40	50	60	minutes
velocity	55	60	70	75	73	40	30	miles/hour



Estimation for the total distance = sum of (rate)(time)

= sum of rectangles

$= \frac{1}{6}(60) + \frac{1}{6}(70) + \frac{1}{6}(75)$

$+ \frac{1}{6}(73) + \frac{1}{6}(40) + \frac{1}{6}(30)$

= 58 miles